

MTH 605: Topology I

Practice Assignment III

1. If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \rightarrow A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.
2. Define an equivalence relation \sim on \mathbb{R}^2 as follows: $(x_0, y_0) \sim (x_1, y_1)$ if $x_0 + y_0^2 = x_1 + y_1^2$. Describe the corresponding quotient space X^* .
3. A *topological group* is a group (G, \cdot) that is also a topological space satisfying the T_1 axiom, such that the group operation $(g, h) \mapsto g \cdot h$ and the map $g \mapsto g^{-1}$ are both continuous maps. Show that $(\mathbb{R}, +)$, $\text{GL}(n)$, and S^1 (seen as a subset of \mathbb{C}) are topological groups.
4. Let G be a topological group, and let H be a subspace and a subgroup of G .
 - (a) Show that both H and \bar{H} are topological groups.
 - (b) Give G/H the quotient topology using the left cosets as partitions. Show that if H is closed in G , then the singletons are closed in G/H .
 - (c) Show that $G \rightarrow G/H$ is open.
 - (d) Show that if H is closed and $H \trianglelefteq G$, then G/H is a topological group.
 - (e) Using (d), show that \mathbb{R}/\mathbb{Z} is a topological group. Describe this space.
5. If τ and τ' be two topologies on X such that $\tau \subset \tau'$. What does the connectedness of X in one topology imply in the other?
6. A space is *totally disconnected* if its only connected subsets are the one-point sets. Show that if X has the discrete topology, then X is totally disconnected.
7. Determine whether the following spaces are connected.
 - (a) An infinite set with the cofinite topology.
 - (b) \mathbb{R}_ℓ .
8. Let $p : X \rightarrow Y$ is a quotient map each of whose fibers are connected. Show that X is connected, whenever Y is connected.
9. Using connectedness, establish the following facts.
 - (a) $(0, 1)$, $(0, 1]$, and $[0, 1]$ are not homeomorphic.
 - (b) \mathbb{R}^n and \mathbb{R} are not homeomorphic for $n > 1$.
10. Show that if $f : [0, 1] \rightarrow [0, 1]$ is a continuous map, then f has a fixed point.